Large Amplitude, Low Frequency Solutions for a Certain Class of Laminar Boundary-Layer Problems

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Theme

A STUDY has been made of the class of unsteady laminar boundary-layer flows for which the velocity at the edge of the boundary layer is given by $u_\delta(x, t) = u_{\delta o}(x)m(t)$. Such flows occur when a nonlifting body moves through a fluid at a varying speed but with fixed orientation with respect to the freestream velocity vector. If the external velocity is of this form both the steady state $[m(t) = 1]$ and the quasi-steady boundary-layer solutions are governed by the same equation. If $m(t)$ is periodic and the reduced frequency is small, the unsteady solution may be obtained as a series expansion, in the reduced frequency, about the quasi-steady solution.

Results are presented for three bodies and it is shown that in each case the unsteady effects are most prominent in the region of adverse pressure gradient. When $m(t)$ is periodic the unsteady shear leads the quasi-steady shear. These results are explained in terms of the local unsteady pressure gradient.

Contents

The present paper is concerned with solutions to the unsteady, incompressible, two-dimensional boundary-layer equations for a certain class of external flows, namely external velocity distributions of the form $u_\delta(x, t) = u_{\delta o}(x)m(t)$. Here $u_\delta(x, t)$ is the unsteady velocity at the upper edge of the boundary layer and $u_{\delta o}(x)$ and $m(t)$ are arbitrary functions of x and time, respectively. Starting with the unsteady boundary-layer equations, which are well known, the analysis involves the scaling of the x and y coordinates, time and the stream function, ψ, according to

$$\xi = x/l, \quad \eta = y[u_\delta(x, t)/vx]^{1/2}, \quad t^* = tU_\infty/l,$$
$$f = \psi(x, y, t)/[u_\delta(x, t)xv]^{1/2}$$

Here ξ is the scaled x coordinate, η is the scaled y coordinate, x and y are the usual boundary-layer coordinates, l is a characteristic length, U_∞ is a characteristic velocity, v is the kinematic viscosity, t^* is the scaled time and f is the scaled stream function. In terms of these new coordinates the momentum equation becomes

$$f''' + \frac{1+M}{2} ff' + M(1-f'^2) + \xi\left(f''\frac{\partial f}{\partial \xi} - f'\frac{\partial f'}{\partial \xi}\right) + \frac{\xi}{u_\delta^*}\frac{\partial u_\delta^*}{\partial t^*}\left[1 - f' - \eta\frac{f''}{2}\right] - \frac{\xi}{u_\delta^*}\frac{\partial f'}{\partial t^*} = 0 \qquad (1)$$

where primes denote differentiation with respect to η and

$$M = \frac{\xi}{u_\delta^*}\frac{\partial u_\delta^*}{\partial \xi}, \qquad u_\delta^* = \frac{u_\delta}{U_\infty}$$

The boundary conditions applicable to Eq. (1) are

$$f'(\xi, o, t^*) = f(\xi, o, t^*) = 0, \qquad \lim_{\eta\to\infty} f'(\xi, \eta, t^*) = 1$$

If the velocity distribution is of the form noted previously the parameter M is a function of x (or ξ) alone. In this case the coefficients of the first four terms in Eq. (1) are functions of x (or ξ) alone and only the coefficients of the last two terms involve time. The last two terms in Eq. (1) are just those terms which result from the time derivatives in the boundary-layer equations. Neglecting these two terms yields the quasi-steady solution for the given external velocity distribution.

Equation (1) with the last two terms dropped is also the equation governing the steady-state boundary layer with the external velocity $u_{\delta o}(x)$. Thus, if the external velocity is of the form prescribed previously, both the steady-state solution and the quasi-steady solution are obtained from the same equation.

To obtain the unsteady solution it is convenient to expand the dimensionless stream function f in a series with coefficients $A_n = (1/m^{n+1})d^n m/dt^{*n}$, i.e.,

$$f(\xi, \eta, t^*) = f_o(\xi, \eta) + A_1(t^*)f_1(\xi, \eta) + A_2(t^*)f_2(\xi, \eta) + \cdots$$
$$[A_1(t^*)]^2 f_{11}(\xi, \eta) + \cdots$$

Although this expansion is quite general, in the special case where m is a periodic function of time the coefficients A_1, A_2, A_3, etc., form a sequence of ascending powers of the dimensionless frequency. Thus this expansion is convenient for investigating low frequency periodic variations in the external velocity. It should be noted that although this analysis is closely related to those of Moore,[1] King,[2] and McCroskey and Dwyer,[3] the expansion parameters employed here involve only time, while that used by the previous authors included both time and the spatial coordinate x.

If the expansion described above is introduced into Eq. (1) there results an infinite set of differential equations for the

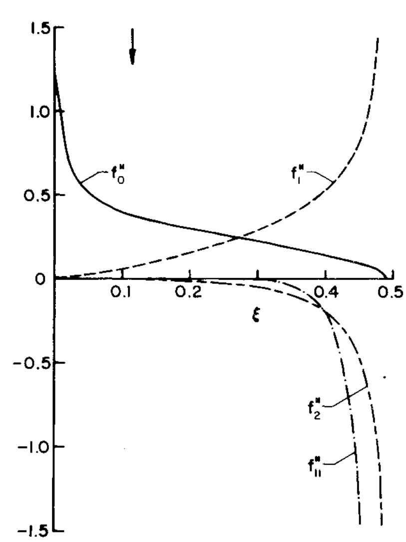

Fig. 1 Components of the wall shearing stress for a 10% thick Joukowski airfoil.

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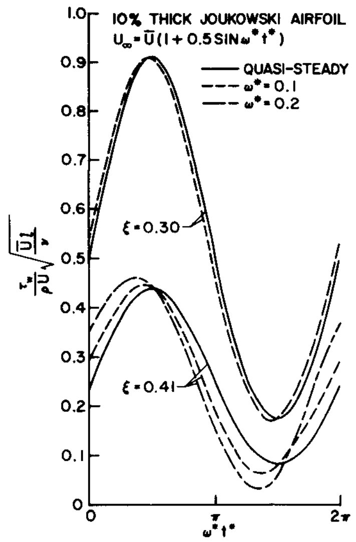

Fig. 2 Dimensionless wall shearing stress variation of two points on a 10% thick Joukowski airfoil.

functions $f_i(\xi, \eta)$. The first four of these equations have been solved, using an implicit finite-difference technique, for three practical two-dimensional bodies: a circular cylinder, a 15% thick Joukowski airfoil and a 10% thick Joukowski airfoil. These bodies are characterized by varying degrees of adverse pressure gradient beyond the pressure minimum.

The significance of the unsteadiness in the flow can be obtained by considering the wall shear, given by

$$\tau_w = \mu(u_\delta^{3/2}/\nu x)^{1/2}[f_o''(\xi,o)+A_1 f_1''(\xi,o)+ A_2 f''(\xi,o) - \cdots A_1^2 f_{11}''(\xi,o)+\cdots]$$

The values of $f_o''(\xi,o)$, $f_1''(\xi,o)$, $f_2''(\xi,o)$ and $f_{11}''(\xi,o)$ are shown in Fig. 1 for the case of a 15% thick Joukowski airfoil. The vertical arrow at the top of the figure indicates the approximate location of the minimum pressure point. In the region of favorable pressure gradient the unsteady contributions to the wall shear, represented by $f_1''(\xi,o)$, $f_2''(\xi,o)$ and $f_{11}''(\xi,o)$, are quite small and a quasi-steady analysis of the boundary layer is quite accurate. Beyond the pressure minimum $f_1''(\xi,o)$ grows quite rapidly and as separation is approached $f_2''(\xi,o)$ and $f_{11}''(\xi,o)$ grow rapidly. These results indicate that unsteady effects will be significant in the region of adverse pressure gradient and particularly as separation is approached.

Up to this point it has not been necessary to specify the form of $m(t)$. We now consider a specific form of $m(t)$, namely $m(t) = 1+\varepsilon \sin \omega t$, which corresponds to a flow in which the body moves with a periodic variation in speed about some mean speed. Figure 2 shows the variation of local skin friction with dimensionless time $\omega^* t^*$ for $\varepsilon = 0.5$, for two stations on a 10% thick Joukowski airfoil and for several values of dimensionless frequency ω^*. For $\xi = 0.3$ the shearing stress is shown for quasi-steady flow and for $\omega^* = 0.1$. At this point the unsteady effects are small but discernible. The point $\xi = 0.41$ is sufficiently far into the region of adverse pressure gradient for the unsteady effects to be sizable. In each case the unsteady wall shear leads the quasi-steady shear, the maximum unsteady wall shear is greater than the maximum quasi-steady wall shear and the minimum unsteady shear is smaller than the minimum quasi-steady wall shear. These effects become more pronounced as the dimensionless frequency is increased.

The effects discussed previously are easily explained in terms of the instantaneous pressure gradient which is made of the "unsteady pressure gradient,"—$\rho(\partial u_\delta/\partial t)$, and the "quasi-steady pressure gradient,"—$\rho u_\delta(\partial u_\delta/\partial x)$; the instantaneous pressure gradient is then

$$\frac{dp}{dx} = -\rho\frac{\partial u_\delta}{\partial t} - \rho_\delta \frac{\partial u_\delta}{\partial x} = -m^2 u_{\delta o}\left(A_1 + \frac{du\delta_o}{dx}\right)$$

As long as A_1 is small compared to $(du_{\delta o}/dx)$ the "quasi-steady pressure gradient" will dominate the flow and the boundary layer will behave as a quasi-steady boundary layer. At the point of minimum pressure $du_{\delta o}/dx$ is zero and the "unsteady pressure gradient" dominates the flow. Beyond the minimum pressure point $du_{\delta o}/dx$ is small (since the laminary boundary layer can sustain only small adverse pressure gradient) and the contribution due to the "unsteady pressure gradient" may be expected to increase, as it does in all cases studied here.

At a fixed chordwise station $u_{\delta o}$ and $du_{\delta o}/dx$ are fixed and the relative magnitude of the "unsteady" pressure gradient depends upon the magnitude of A_1. For the periodic form of $m(t)$ assumed previously, A_1 is positive in the first and fourth quadrants of $\omega^* t^*$ and negative in the second and third. Thus if the "quasi-steady pressure gradient" is adverse the effect of the "unsteady pressure gradient" is to make the instantaneous pressure gradient less adverse in the first and fourth quadrants, leading to greater wall shear, and more adverse in the second and third quadrants, leading to lower wall shear. These are just the effects shown in Fig. 2.

References

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